Radiosity Equation

Assume all surfaces are Lambertian diffuse:

$$L(x' \to x'') = L_e(x' \to x'') + \int_S f_r(x \to x' \to x'')\, L(x \to x')\, V(x,x')\, G(x,x')\, dA$$

$$L(x' \to x'') = L_e(x' \to x'') + f_r(x') \int_S L(x \to x')\, V(x,x')\, G(x,x')\, dA$$

$$\frac{B(x')}{\pi} = \frac{E(x')}{\pi} + \frac{\rho(x')}{\pi} \int_S \frac{B(x)}{\pi} V(x,x')\, G(x,x')\, dA$$

$$B(x') = E(x') + \rho(x') \int_S B(x) \frac{G(x,x')\, V(x,x')}{\pi}\, dA$$

Radiosity Equation

The radiosity function space is infinite

- An infinite number of values is required to specify the function
- Solving the radiosity equation at one point tells us nothing about neighboring points

Solving the radiosity equation requires solving for the exact functional form, or computing the radiosity at infinitely many points

Finite Element Methods

Approximate a function by a linear combination of basis functions

$$B(x) \approx \hat{B}(x) = \sum_{i=1}^{n} B_i N_i(x)$$

The basis functions, $N_i(x)$, have finite support

Constant basis functions:

$$N_i(x) = \begin{cases} 0 & \text{if } x \text{ is outside element } i \\ 1 & \text{if } x \text{ is inside element } i \end{cases}$$

Error Metrics

We need a measure of the error

True error: $\varepsilon(x) = B(x) - \hat{B}(x)$

Impractical, since we must know the solution

Instead substitute the approximation into the radiosity equation and find the residual

$$\hat{B}(x) \approx E(x) + \rho(x) \int_S \hat{B}(x') \frac{G(x,x')\,V(x,x')}{\pi}\, dA$$

$$r(x) = \hat{B}(x) - E(x) - \rho(x) \int_S \hat{B}(x') \frac{G(x,x')\,V(x,x')}{\pi}\, dA$$

Error Metrics

The residual is equivalent to applying the operator K to the negation of the error

$$-K\varepsilon(x) = K\hat{B}(x) - KB(x)$$

$$= \hat{B}(x) - E(x) - \rho(x)\int_S \hat{B}(x')\frac{G(x,x')\,V(x,x')}{\pi}\,dA$$

Where

$$Kb(x) = b(x) - \rho(x)\int_S b(x')\frac{G(x,x')\,V(x,x')}{\pi}\,dA$$

$$KB(x) = E(x)$$

Exact solution yields a residual of zero

Error Metrics

We approximate the magnitude of the residual by a linear sum of finite support functions (weighting functions)

Known as the weighted residual method

$$|r(x)| = \sum_{i=1}^{n} \left| \langle r(x), W_i(x) \rangle \right|$$

We minimize the residual norm by finding a solution that makes the n terms zero

What weighting functions to use?

Point Collocation

Point sample using delta functions

$$W_i(x) = \delta(x - x_i)$$

Requires the solution to be exact at a specified number of points.

The residual can be nonzero between these points

Choosing the points to be the locations of the nodes in the radiosity solution is known as point collocation

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Point Collocation

This results in n simultaneous constraints:

$$\hat{B}(x_i) - E(x_i) - \rho(x_i) \int_S \hat{B}(x') \frac{G(x_i, x')\, V(x_i, x')}{\pi}\, dA = 0$$

Substituting for $\hat{B}$:

$$\sum_{j=1}^{n} B_j N_j(x_i) - E(x_i) - \rho(x_i) \int_S \sum_{j=1}^{n} B_j N_j(x') \frac{G(x_i, x')\, V(x_i, x')}{\pi}\, dA = 0$$

Factoring the unknown B_j:

$$\left[\sum_{j=1}^{n} B_j \left[N_j(x_i) - \rho(x_i) \int_S N_j(x') \frac{G(x_i, x')\, V(x_i, x')}{\pi}\, dA \right] \right] - E(x_i) = 0$$

Point Collocation

Expressing this as matrix-vector form:

$$\left[\sum_{j=1}^{n} B_j \left[N_j(x_i) - \rho(x_i) \int_S N_j(x') \frac{G(x_i, x') V(x_i, x')}{\pi} \, dA \right]\right] - E(x_i) = 0$$

$$KB = E$$

Where

$$K_{ij} = N_j(x_i) - \rho(x_i) \int_S N_j(x') \frac{G(x_i, x') V(x_i, x')}{\pi} \, dA$$

K depends only on the geometry and materials in the scene

K can be precalculated for each scene

Galerkin Method

Force the weighted integrals to zero:

$$\langle W_i(x), r(x) \rangle = \int_S W_i(x) r(x) \, dA = 0$$

Substituting the residual:

$$0 = \int_S W_i(x) \hat{B}(x) \, dA - \int_S W_i(x) E(x) \, dA$$

$$- \int_S W_i(x) \rho(x) \int_S \hat{B}(x') \frac{G(x, x') V(x, x')}{\pi} \, dA' \, dA$$

Galerkin Method

Use the same weighting and basis functions:

$$W_i(x) = N_i(x)$$

Substituting gives:

$$0 = \int_S N_i(x)\hat{B}(x)\,dA - \int_S N_i(x)E(x)\,dA$$

$$-\int_S N_i(x)\rho(x)\int_S \hat{B}(x')\,\frac{G(x,x')V(x,x')}{\pi}\,dA'\,dA$$

Galerkin Method

Collecting the unknowns:

$$\sum_{j=1}^{n} B_j\left[\int_S N_i(x)N_j(x)\,dA - \int_S N_i(x)\rho(x)\int_S N_j(x')\,\frac{G(x,x')V(x,x')}{\pi}\,dA'\,dA\right]$$

$$-\int_S E(x)N_i(x)\,dA = 0$$

Expressed in matrix form:

$$KB = E$$

$$K_{ij} = \int_S N_i(x)N_j(x)\,dA - \int_S N_i(x)\rho(x)\int_S N_j(x')\,\frac{G(x,x')V(x,x')}{\pi}\,dA'\,dA$$

$$E_i = \int_S E(x)N_i(x)\,dA$$

Constant Radiosity Elements

Using constant basis functions:

$$\int_S N_i(x)N_j(x)\,dA = \delta_{ij}A_i$$

$$\int_S E(x)N_i(x)\,dA = E_i A_i$$

$$\int_S N_i(x)\rho(x)\int_S N_j(x')\frac{G(x,x')V(x,x')}{\pi}\,dA'\,dA = \rho_i\int_{A_i}\int_{A_j}\frac{G(x,x')V(x,x')}{\pi}\,dA_j\,dA_i$$

Substituting gives:

$$\left[\sum_{j=1}^{n} B_j\left[\delta_{ij}A_i - \rho_i\int_{A_i}\int_{A_j}\frac{G(x,x')V(x,x')}{\pi}\,dA_j\,dA_i\right]\right] - E_i A_i = 0$$

Constant Radiosity Elements

Simplifying:

$$\left[\sum_{j=1}^{n} B_j\left[\delta_{ij}A_i - \rho_i\int_{A_i}\int_{A_j}\frac{G(x,x')V(x,x')}{\pi}\,dA_j\,dA_i\right]\right] - E_i A_i = 0$$

$$\sum_{j=1}^{n} B_j\left[\delta_{ij} - \frac{\rho_i}{A_i}\int_{A_i}\int_{A_j}\frac{G(x,x')V(x,x')}{\pi}\,dA_j\,dA_i\right] = E_i$$

$$\sum_{j=1}^{n} B_j\left[\delta_{ij} - \rho_i F_{ij}\right] = E_i$$

Rearranging gives the classic radiosity equation:

$$\boxed{B_i = E_i + \rho_i\sum_{j=1}^{n} B_j F_{ij}}$$

Interpreting the Radiosity Equation

F_{ij} are known as Form Factors:

$$F_{ij} = \frac{1}{A_i} \int_{A_i} \int_{A_j} \frac{G(x,x')V(x,x')}{\pi} \, dA_j \, dA_i$$

Form Factors represent the fraction of energy leaving element i that arrives directly at element j

Reciprocity of Form Factors:

$$F_{ij} A_i = F_{ji} A_j$$

CS174 Winter 00 Lecture 5

Interpreting the Radiosity Equation

Converting to a more physically intuitive form:

$$B_i = E_i + \rho_i \sum_{j=1}^{n} B_j F_{ij}$$

$$B_i A_i = E_i A_i + \rho_i \sum_{j=1}^{n} B_j F_{ij} A_i$$

$$B_i A_i = E_i A_i + \rho_i \sum_{j=1}^{n} F_{ji} B_j A_j$$

The power leaving element i, $B_i A_i$, is equal to the power emitted, $E_i A_i$, plus the power reflected.

The power reflected depends on the power arriving at i directly from all other surfaces, $F_{ji} B_j A_j$, and the surface reflectivity, ρ_i .

CS174 Winter 00 Lecture 5

Matrix Solutions

The radiosity equation can be expressed as:

$$K\,B = E$$

$$[M - PF]B = E$$

where

M is the matrix of basis function inner products

P is the diagonal matrix of element reflectivities

F is the matrix of form factors

M is the identity matrix for constant elements

Matrix Properties

Size:

Generally, n x n (square) where n is the number of basis functions

Symmetry:

Generally, not symmetric

We can induce a symmetry by multiplying the *ith* row by A_i $(F_{ij}A_i = F_{ji}A_j)$

We can now use solvers requiring symmetry

Matrix Properties

Sparsity:

Generally, the matrix is full (not sparse)

$K_{ij} = 0$ iff:

$\rho_i = 0$ (surface is completely black)

or

$F_{ij} = 0$ (elements are completely invisible)

Sometimes, there are independent sets of elements (2 separate rooms) – the matrix is then block diagonal and separable

Matrix Properties

Diagonal Dominance:

The magnitude of the diagonal element is greater than or equal to the sum of the magnitudes of the off-diagonal elements

$$\sum_{j=1,\, j \neq i}^{n} \left| K_{ij} \right| \leq \left| K_{ii} \right|, \forall i$$

For constant radiosity elements, the diagonal is 1, and the sum of the reflectivity weighted form factors must be less than or equal to 1

Ensures convergence for many iterative solvers

Matrix Properties

Spectral radius:

- Indicates the speed of convergence
- As reflectivities increase, slower convergence

Condition:

- Describes how sensitive the solution is to small perturbations in the input
- Radiosity matrices are usually well conditioned

Iterative Solutions

Jacobi Iteration

Update one entry assuming all others are correct:

$$B_i^{(k+1)} = E_i - \sum_{j \neq i} K_{ij} \frac{B_j^{(k)}}{K_{ii}}$$

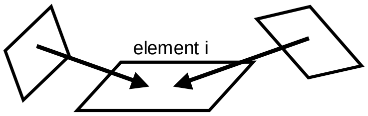
Gathering

Gauss-Seidel

Use the most updated radiosities:

$$B_i^{(k+1)} = E_i - \sum_{j=1}^{i-1} K_{ij} \frac{B_j^{(k+1)}}{K_{ii}} - \sum_{j=i+1}^{n} K_{ij} \frac{B_j^{(k)}}{K_{ii}}$$

Update a single radiosity by summing the contributions of all other radiosities weighted by the reflectivities and form factors (in row i)



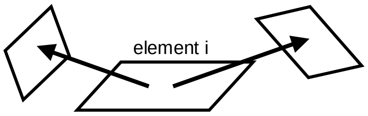
Equivalent to gathering light from all other elements

Shooting

Update using the columns rather than the rows

$$B_j += \Delta B_i \rho_j F_{ji}, \forall j$$

Update all radiosities due to a single element's radiosity weighted by the reflectivities and form factors (in column i)



Equivalent to shooting light to all other elements

Progressive Refinement

Shoot from the element with the most unshot radiosity

```
Progressive Refinement
  forall i
    $B_i$ = $delB_i$ = $E_i$
  while(not converged)
    find i such that $delB_i*A_i$ is largest
    forall elements j
      delRad = $delB_i*rho_i*F_{ji}$
      $delB_j$ = $delB_j$ + delRad
      $B_j$ = $B_j$ + delRad
    $delB_i$ = 0
    display the image using the current values of $B_i$
```

Ambient Term

Solution proceeds from dark to light

Add a term approximating the unshot radiosity

1) Compute Estimated Form Factor

$$F_{ij}^{estimated} = \frac{A_j}{\sum_{k=1}^{n} A_k}$$

Equal to the fraction of the environment taken up by element j

Ambient Term

2) Compute the average reflectivity of the scene

$$\rho^{average} = \frac{\sum_{i=1}^{n} \rho_i A_i}{\sum_{i=1}^{n} A_i}$$

3) Compute the overall reflection factor

$$\mu = 1 + \rho_{average} + \rho^2_{average} + \ldots = \frac{1}{1 - \rho_{average}}$$

Reflectivites for 0, 1, 2, …, m bounces

Ambient Term

4) Add the ambient term to the display radiosities

$$I_a = \mu \sum_{j=1}^{n} \Delta B_j \ F_{ij}^{estimated}$$

The ambient term is used only for display

The ambient term can be incrementally updated

Form Factor Calculation

Expensive to compute

- Must deal with all other elements in the scene
- Must deal with complex shape relationships

Expensive to store

- A scene containing 1E5 elements with 90% sparsity (most elements cannot see each other) requires 4E9 bytes (4 GB) of storage (at 4 bytes per form factor)

Form Factor Calculation

The fraction of energy leaving element i that arrives directly at element j

$$F_{ij} = \frac{1}{A_i} \int_{A_i} \int_{A_j} \frac{G(x_i, x_j)\, V(x_i, x_j)}{\pi}\, dA_j\, dA_i$$

$$= \frac{1}{A_i} \int_{A_i} \int_{A_j} \frac{\cos\theta_i \cos\theta_j\, V(x_i, x_j)}{\pi \left| x_i - x_j \right|^2}\, dA_j\, dA_i$$

Differential Form Factor:

$$F_{dA_i dA_j} = \frac{G(x, x')V(x, x')}{\pi}\, dA_j\, dA_i$$

Form Factor Properties

Reciprocity of Form Factors:

$$F_{ij} A_i = F_{ji} A_j$$

In a closed environment:

$$\sum_{j=1}^{n} F_{ij} = 1$$

For a convex (or flat) surface:

$$F_{ii} = 0$$

Form Factor - Numerical

Convert area integral into a contour integral:

$$F_{ij} = \frac{1}{2\pi A_i} \oint_{C_j} \oint_{C_i} \ln(r)\,dx_i dx_j + \ln(r)\,dy_i dy_j + \ln(r)\,dz_i dz_j$$

Computationally expensive

Doesn’t account for visibility

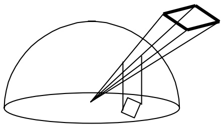
- Can be augmented to recognize visibility by altering the contours to account for occlusion

Form Factor - Hemicube

If the distance between the two patches is large:

$$F_{ij} \approx F_{dA_i A_j} = \int_{A_j} \frac{\cos\theta_i \cos\theta_j V(x_i, x_j)}{\pi \left| x_i - x_j \right|^2} dA_j$$

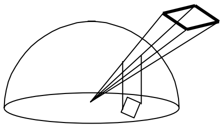
Nusselt Analogue: the form factor is equal to the fractional area formed by the projection of the patch onto the hemi circle



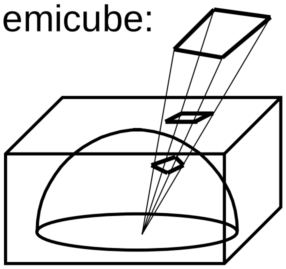
Any patch with the same projection has the same form factor

Form Factor - Hemicube

Any patch with the same projection on the sphere has the same form factor:



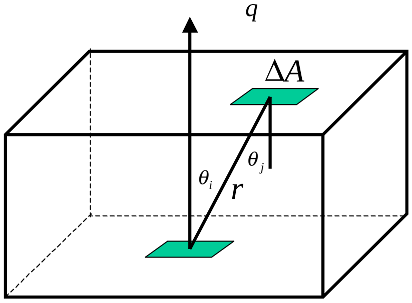
Can also use a hemicube:



Form Factor - Hemicube

1) Center a hemicube around element i
2) Project all other elements onto the hemicube

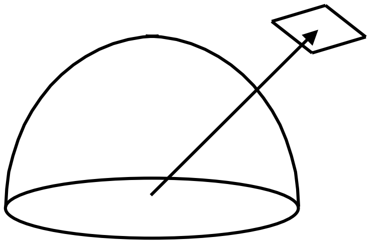
$$F_{ij} = \sum_q \Delta F_q \qquad q \in \text{all element j pixels}$$



$$\Delta F_q = \frac{\cos\theta_i \cos\theta_j}{\pi r^2} \Delta A$$

Form Factor - Hemisphere

1) Center a hemisphere around element i
2) Trace a ray through each pixel
   a) Find the closest intersection
   b) Add a delta form factor to the intersected element's



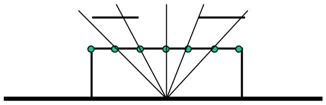
$$F_{ij} += \Delta F$$

This algorithm is highly parallelizable

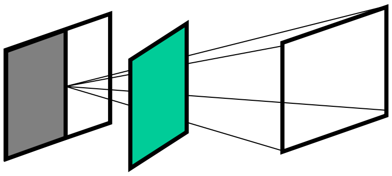
Form Factor - Hemicube/sphere

Problems

- Aliasing



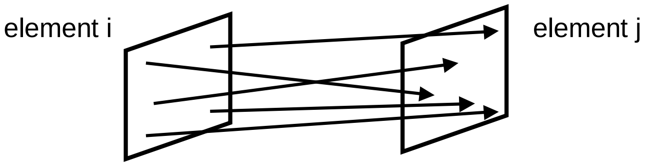
- Accuracy depends on the distance between the elements
- Intervening patches may be missed



Form Factor – Ray Tracing

Compute the integral stochastically by tracing rays

$$F_{ij} = \frac{1}{A_i} \int_{A_i} \int_{A_j} \frac{\cos\theta_i \cos\theta_j \, V(x_i, x_j)}{\pi \left| x_i - x_j \right|^2} \, dA_j \, dA_i$$



Many rays must be cast for accuracy

Form Factor – Hybrid

Hemicube breaks when the distance is small

$$F_{ij} \not\approx F_{dA_i A_j}$$

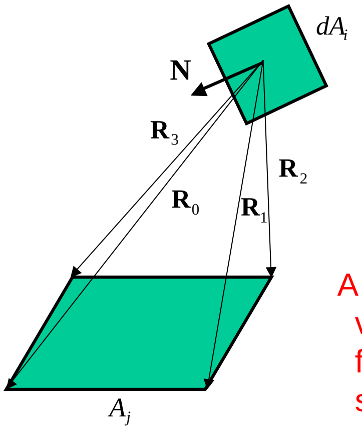
When the distance is small, calculate:

$$F_{ij} = \frac{1}{A_i} \int_{A_i} \int_{A_j} \frac{\cos\theta_i \cos\theta_j \, V(x_i, x_j)}{\pi \left| x_i - x_j \right|^2} \, dA_j \, dA_i$$

Can use any accurate calculation technique

Form Factor – Polygonal

Unoccluded differential-to-finite area form factor:

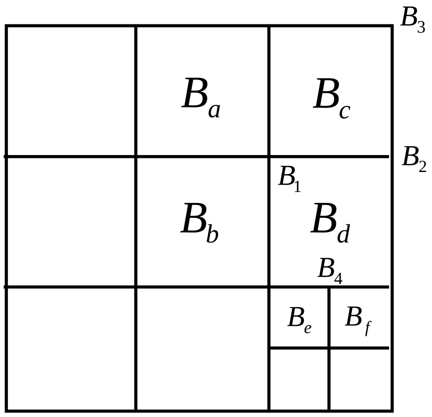


$$F_{dA_i A_j} = -\frac{1}{2\pi} \sum_k \mathbf{N} \cdot \mathbf{E}_k$$

$$\mathbf{E}_k = \cos^{-1}\left(\mathbf{R}_k \cdot \mathbf{R}_{k\oplus 1}\right) \frac{\mathbf{R}_k \times \mathbf{R}_{k\oplus 1}}{\left\| \mathbf{R}_k \times \mathbf{R}_{k\oplus 1} \right\|}$$

A visibility coefficient can be computed via ray tracing and multiplied onto the form factor. If visibility is not constant subdivide and recompute.

Computing Vertex Colors



$$B_1 = \frac{B_a + B_b + B_c + B_d}{4}$$

$$B_2 = \frac{-B_a - B_b + 7B_c + 7B_d}{4}$$

$$B_3 = \frac{-B_a - B_b + 7B_c - B_d}{4}$$

$$B_4 = \frac{2B_d + B_e + B_f}{4}$$

Complete Radiosity Solution